British Mathematical Olympiad

Round 2

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Solutions

1. In the sequence 7, 76, 769, 7692, 76923, 769230, . . ., the nth term is given by the first n digits after the decimal point in the expansion of $10/13 = 0.7692307692\ldots$.

Prove that of the first 60 terms of the sequence, at least 49 have three or more prime factors (repeated prime factors are allowed; for example, $76 = 2 \times 2 \times 19$ has three prime factors).

Solution

In the sequence 76, 76923076, 76923076923076, . . ., every term is a multiple of 4, so has at least 3 prime factors. The same is true for the terms ending in 2 or ending in 0. This gives 30 terms with 3 prime factors.

The sum of the six digits 76923 is 18, so every term ending in 3 is a multiple of 9, giving 10 more terms with 3 prime factors.

Finally, look at terms ending in 7. Every such term is a multiple of 7 (since $76923 = 7 \times 10989$). Moreover, multiplying by 13, we get a number

$$999\ldots91 = 10^{6n+4} - 9 = (10^{3n+2} - 3)(10^{3n+2} + 3)$$

One of the two brackets must be a multiple of 7, expressing the number as a product of 3 factors for $n > 0$. This gives an extra 9 terms with 3 prime factors.

Remark

2. Find all functions f from the integers to the integers such that for all integers n:

$$2f(f(n)) = 5f(n) - 2n.$$

SOLUTION

Clearly $f(n) = 2n$ is a solution. We claim there are no others.

We have $f(n) - 2n = 2(f(f(n)) - 2(f(n))$, so $2^1 \mid (f(n) - 2n)$.

Now suppose $2^k \mid ((f(n) - 2n))$. Replacing $n = f(n)$ and doubling gives us

$2^{k+1} \mid 2f(f(n)) - 4f(n) = f(n) - 2n.$

Thus $f(n) - 2n$ has arbitrarily large factors for any fixed n which is enough.

ALTERNATIVE

Define $g(n) = f(n) - 2n$ then the equation becomes:

$$g(n) = 2g(2n + g(n)) \qquad (*)$$

Assume, for contradiction, that there exists m with $g(m) \neq 0$. Choose the n with $v_2(g(m))$ minimal.

Observe that $(*)$ forces $g(n)$ to be even for all n. If $g(n) \neq 0$ then this means $v_2(g(n)) \geq 1$. Hence, setting $n = m$ in $(*)$:

$$0 \neq g(m) = 2\underbrace{g(2m + g(m))}_{\neq 0} \Rightarrow v_2(g(2m + g(m))) = v_2(g(m)) - 1 < v_2(g(m)) \text{ and } g(2m + g(m)) \neq 0$$

But this contradicts our assumption about minimality of $v_2(g(m))$. Thus $g \equiv 0$ and $\boxed{f(n) = 2n}$ for all n which is indeed a solution.

ALTERNATIVE

Fix $m \in \mathbb{Z}$ and define $a_i = f^{(i)}(m)$ (that is f applied i times to n).

By induction using $2a_{i+2} = 5a_{i+1} - 2a_i$ (from setting $n = f^{(i)}(m)$ in the original equation):

$$a_i = \frac{4m - 2f(m)}{3 \cdot 2^i} + \frac{2^i}{3}(2f(m) - m)$$

As $a_i \in \mathbb{Z}$, $\frac{1}{3 \cdot 2^i} \xrightarrow{i \to \infty} 0$ and the second term is at most $\frac{1}{3}$ away from an integer, the first term must vanish which forces $f(m) = 2m$ which, as we saw before, is a solution.

REMARK

The original wording of this problem was different. Here it is:

Original wording Point P lies outside circle Γ and the tangents from P to Γ touch it at T_1 and T_2. The midpoints of PT_1 and PT_2 are M_1 and M_2 respectively. A point Q on Γ lies on the other side of T_1T_2 to P. The lines QT_1 and QT_2 meet line M_1M_2 at R_1 and R_2 respectively. Prove that PR_1QR_2 is cyclic.

3. Let ABC be an acute-angled triangle with $AB > AC$. Let P be the intersection of the tangents to the circumcircle of ABC at B and C. The line through the midpoints of line segments PB and PC meets lines AB and AC at X and Y respectively.

Prove that the quadrilateral $AXPY$ is cyclic.

Solution

Using the AST and the parallel lines T_1T_2 and M_1M_2 we can establish the following similar triangles:

$QT_1T_2 \sim M_1R_1T_1 \sim M_2T_2R_2$.

Using the midpoints we see that $R_1M_1/M_1P = PM_2/M_2R_2$ so triangles R_1M_1P and PM_2R_2 are similar since PM_1M_2 is isosceles.

Thus $\angle M_1PR_1 + \angle R_2PM_2 = \angle TM_1R_1 = \angle M_1T_1T_2 = \angle T_1QT_2$. The first equality comes from viewing $\angle T_1M_1R_1$ as an external angle of R_1M_1P and the others from parallel lines and AST.

It is now straightforward to show that $R_1QR_2 + R_2PR_1 = \pi$ as required, by using the isosceles triangle PT_1T_2.

Alternative

Angle-chasing and using AST we have:

$$\angle T_1M_1R_1 = \angle PM_1M_2 = \angle PT_1T_2 = \angle T_1QT_2$$
$$\angle R_1T_1M_1 = 180^\circ - \angle PT_1Q = \angle QT_2T_1$$

so $\triangle R_1T_1M_1 \sim \triangle T_1T_2Q$.

Let N be the midpoint of T_1T_2 then, using the above, we have:

$$\angle QT_2N = \angle R_1T_1M_1 \quad \text{and} \quad \frac{QT_2}{NT_2} = 2 \cdot \frac{QT_2}{T_1T_2} = 2 \cdot \frac{T_1M_1}{T_1R_1} = \frac{T_1P}{T_1R_1}$$

so $\triangle QT_2N \sim \triangle PT_1R_1$.

Finally, using properties of symmedians and the above similarities:

$$\angle PR_1R_2 = \angle PR_1T_1 - \angle M_1R_1T_1 = \angle T_2NQ - \angle T_2T_1Q = \angle T_1QN = \angle PQR_2$$

which shows PR_1QR_2 cyclic.

4. Let $m < n$ be positive integers. Start with n piles, each of m objects. Repeatedly carry out the following operation: choose two piles and remove n objects in total from the two piles. For which (m, n) is it possible to empty all the piles?

Solution

It is possible whenever $m = n - k$ for some positive integer $k \mid n$.

Firstly: it is possible in this case. Select k piles and subdivide them into piles of size k. This results in $n - k$ piles of size k and $n - k$ piles of size $n - k$, from where it is easy to see how to empty all the piles. While the subdivision move isn't mentioned in the rules, it's clear that the sequence of operations on the subdivided piles would also work on the original piles.

To show that it is not possible in other cases, let $g = \gcd(n, m)$. Consider the graph whose vertices are piles and whose edges are between pairs of piles that are ever chosen together. Note that this is a true graph as $m < n$ so there can't be any cases where we select the same pair twice. The graph is not necessarily connected, but any maximal connected subgraph must have at least $\frac{n}{g}$ vertices in order that the sum of pile sizes is a multiple of n. There are therefore at most g such disconnected subgraphs, which means that there must be at least $n - g$ edges. After all piles are empty, there must have been m operations carried out, in other words $m \geq n - g$. Since g is the gcd, the only possibility is $m = n - g$, as required.

For the alternative: it is always possible when $m \geq n$, so the set of solutions is as previously plus all cases with $m \geq n$. In particular, it is always possible if the total number of objects is at least n times the number of non-empty piles, and of course a multiple of n.

Prove this separately for each n via two-dimensional induction on the number of non-empty piles p and the total number of objects s across all piles. We need to prove that we can win whenever $s \geq np$, with $n \mid s$. The smallest case is $p = 1$ and $s = n$, which is trivial. Now assume that it is true for all numbers of piles $p' \leq p$ and all objects counts $s' \leq s$, as long as $s' >= p'n$ (excluding the case in question, of course, ie at least one of the inequalities must be strict). Since $s \geq np$, the largest pile must be at least n objects. Consider the smallest pile. If it is no more than size n, then we can remove all objects from that pile, and enough from the largest to complete the removal of n objects. This reduces p by one and s by n, so the conditions are still satisfied, and the result now holds by the IH. If, on the other hand, the smallest pile is larger than n, we must have $s > pn$, ie $s \geq pn + n$. In this case, remove n objects from any pile, resulting in a smaller position, and again we can use the IH. So either way the result holds.

This proves that we can succeed whenever $m \geq n$, as well as for the given cases where $m < n$, and no others.

Alternative

Let $n = m + k$. As in the solution above, if $k \mid n$ then this is possible so assume from now $k \nmid n$. We work $\pmod{k}$ (so when we say we remove s stones from a pile, we really mean any number of stones s' such that $s' \equiv s \pmod{k}$).

Each move removes n stones from a pair of piles. There are two types of move:

- **Type A:** We remove n stones from one pile (assign it score 1) and 0 stones from another pile (assign it score 0).
- **Type B:** We remove $s \not\equiv n \pmod{k}$ stones from one pile and $n - s$ stones from the other pile (assign each pile a score of $\frac{1}{2}$).

Note that to get each pile to be $0 \pmod{k}$, its total score must be at least 1 (as $k \nmid n$, the piles don't start off at $0 \pmod{k}$). This shows that at the end the total score must be $\geq n$. But each move contributes 1 to the total score and there are m moves thus the total score is m meaning $m \geq n$ which contradicts the condition $n > m$.